Western Hemisphere Colloquium on Geometry & Physics

Hall-Littlewood Chiral Rings & Derived Higgs Branches

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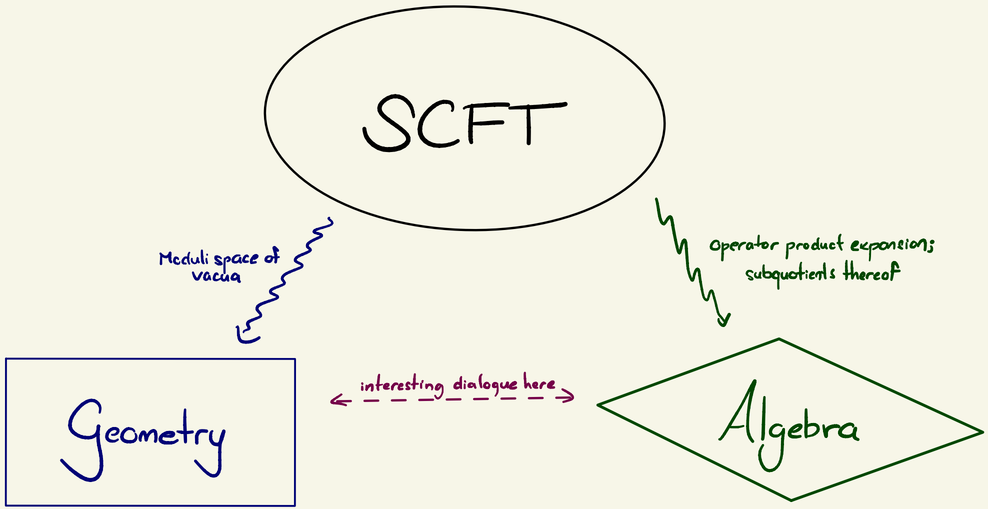
Context

Supersymmetric, and especially superconformal, quantum field theories are algebraically & geometrically rich objects.



Many interesting things to study on both sides of this dichotomy. Ideally should enrich each other.

Today I'll talk about one development in this area that nicely illustrates this interplay.

Summary & Overview

Today I'm talking about 4d $\mathcal{N}=2$ SCFTs and their Higgs branch of vacua $\mathcal{M}_H$.

From the SCFT perspective, it is natural to study $\mathcal{M}_H$ as an affine variety (in terms of ring of functions R_H).

I will point out that in some sense, this is not quite the "right" object to study.

Better is the Hall-Littlewood chiral ring.

Geometrically, this turns out to mean treating $\mathcal{M}_H$ as a derived scheme.

Physically, corresponds (roughly) to keeping track of residual Abelian gauge symmetry in generic Higgs branch vacua.

Some observations:

- R_{HL} enjoys extra symmetries.
- Hints of a relationship with structure of full moduli space of vacua, but some subtleties remain.
- Interesting relation to vertex algebras (v. technical point).

Outline

- 4d $\mathcal{N}=2$ SUSY, Moduli Spaces, and Higgs branches.
- Hall-Littlewood index & chiral ring
- Hall-Littlewood cohomology & derived symplectic reduction
- Geometric & physical aspects of R_{HL}
- Examples: class $\mathcal{S}$ of type A_1
- Concluding remarks

4d $\mathcal{N}=2$ SCFT Primer (Lagrangian perspective)

Lagrangian theories constructed using two types of supermultiplets:

- Vector multiplet $\{\phi, \lambda^{I=1,2}_\alpha, A_\mu\}$ in adjoint rep. of gauge group G.
- Hypermultiplets $\{q, \psi_\alpha\}$ in quaternionic G-rep: R (often $R = R_{cx} \oplus \overline{R_{cx}}$)

Conformal invariance restricts (G, R):*

- Semi-simple gauge group $G = \underbrace{G_1 \times G_2 \times \cdots \times G_n}_{\text{simple factors}}$
- vanishing beta-functions $2h^\vee(G_i) = \sum_\alpha n_{R_\alpha} c_2(R_\alpha)$ (quaternionic G_i-irrep; n_{R_α}: multiplicity of R_α)
- only freedom in Lagrangian is values of cx. gauge couplings.

In non-Lagrangian theories, restriction on $c_2(R_\alpha)$ replaced by restriction on $\langle u(1)_r G_i G_i \rangle$ – triangle anomaly.

* additional constraint for $USp(2n)$ gauge groups due to global anomalies.

4d $\mathcal{N}=2$ SCFT Primer (Lagrangian perspective)

My focus will be on the "moduli spaces of vacua" for these theories. These can be quite intricate.

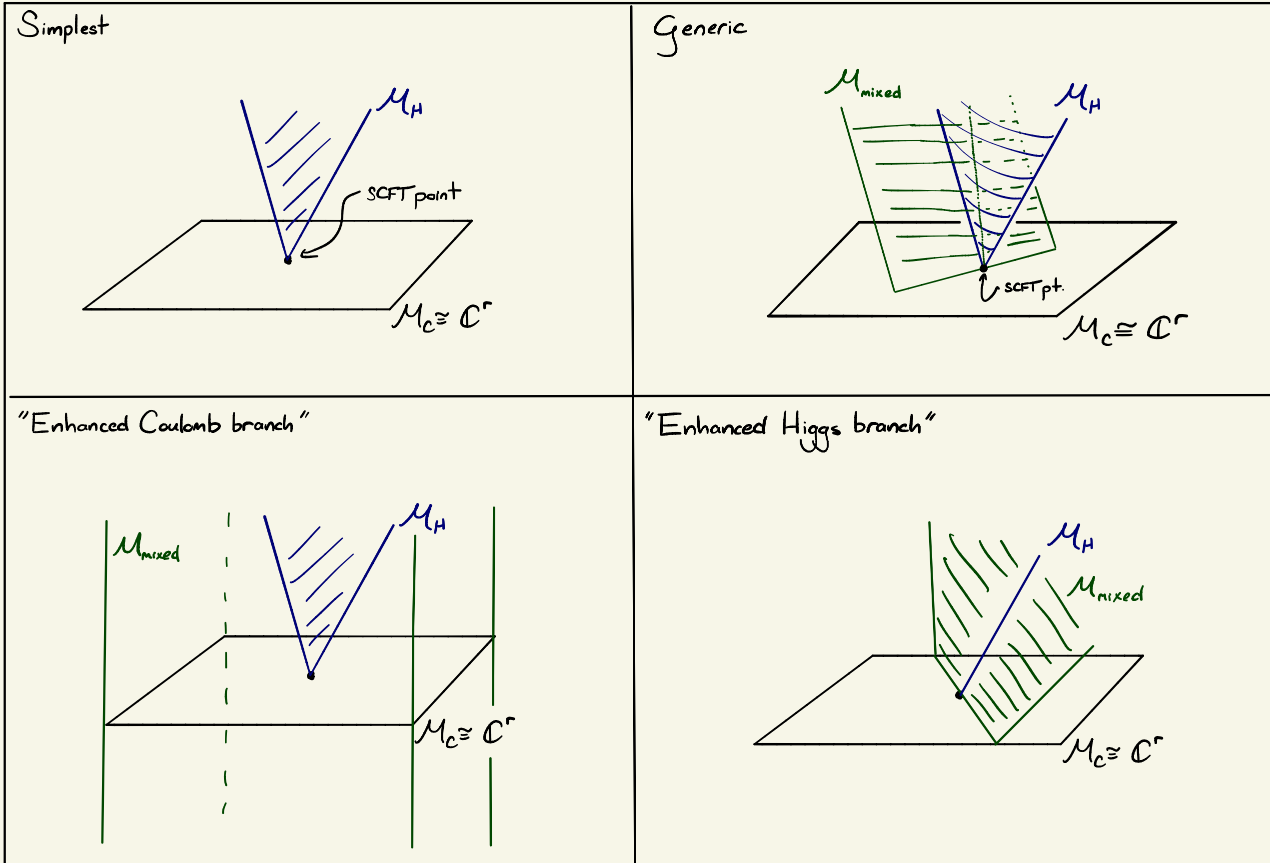
Full moduli space is partitioned into "branches":

- **Coulomb branch:** $\langle\phi\rangle \neq 0$, $\langle q\rangle = 0$
  $U(1)_r$ spontaneously broken
  $SU(2)_r$ unbroken
  — algebraically, $\mathcal{M}_c \cong \mathbb{C}^r$ $(r = \operatorname{rank} \mathcal{G})$
  metric has interesting coupling dependence
  special-Kähler

- **Higgs branch:** $\langle q\rangle \neq 0$, $\langle\phi\rangle = 0$
  $SU(2)_r$ spontaneously broken
  $U(1)_r$ unbroken
  — (conical) hyperkähler space
  algebraically intricate
  metric classically exact

- **Mixed branch:** $\langle q\rangle \neq 0$, $\langle\phi\rangle \neq 0$
  $SU(2)_r$ spontaneously broken
  $U(1)_r$ spontaneously broken
  locally product of "Higgs" & "Coulomb"
  — not so much work here

4d $\mathcal{N}=2$ Moduli Spaces (cartoons)



Higgs branches (Lagrangian perspective)

Higgs branches for gauge theories are realised as hyperkähler (HK) quotients.

Let V_R be vector space carrying quaternionic $\mathcal{G}$-rep R:

- HK moment map $\mu : V_R \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$ (i.e., triplet of real moment maps)
- HK quotient $\mathcal{M}_H \cong V_R /\!/\!/ G := \mu^{-1}(0) / G$ (D-terms & F-terms $= 0$ / gauge)
- More generally, for theory w/ Higgs branch $\mathcal{M}_H \circlearrowleft G$, can gauge G: $\mathcal{M}_H^G = \mathcal{M}_H /\!/\!/ G$

From perspective of fixed cx. str / $\mathcal{N}=1$ subalgebra, these are holomorphic-symplectic singularities (non-deg. (2,0)-form on smooth locus).
Gauging becomes (cx.)-symplectic quotient.

- $\mu_{\mathbb{C}} = \mu_2 + i\mu_3$ is holomorphic moment map for $\mathcal{G}_{\mathbb{C}}$ action w.r.t. holo.-symplectic form.
- $\mathcal{M}_H \cong (\mu_{\mathbb{C}}^{-1}(0)) /\!/ \mathcal{G}$

> Remark: in SCFTs, 0 is always an irregular value of $\mu_{\mathbb{C}}$, so result is stratified (holo.) symplectic space.

4d $\mathcal{N}=2$ SCFT Primer (algebraic perspective)

Primary protagonist: (Complexified) superconformal algebra: $\mathfrak{su}(4|2)$

Cartan generators $\{\Delta, j_1, j_2, R, r\}$

- ▷ Bosonic (even) subalgebra: $\mathfrak{su}(4) \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r \supset \mathfrak{so}(1,1)_\Delta \times \mathfrak{su}(2)_1 \times \mathfrak{su}(2)_2 \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$

- ▷ Fermionic (odd) generators: $\{Q^I_\alpha, \tilde{Q}^I_{\dot\alpha}, S_I^\alpha, \tilde{S}_I^{\dot\alpha}\}$

  $\mathfrak{su}(2)_1$ Doublet, $\mathfrak{su}(2)_2$ Doublet, $\mathfrak{su}(2)_R$ Doublet

Local operators organised into unitary irreps (w.r.t. appropriate reality/Hermiticity conditions)

$$\boxed{\begin{tikzcd} & \mathcal{O}_{\Delta,R,r,j_1,j_2}(x) \arrow[dl, "Q"'] \arrow[d, "R_I{}^J \,|\, M_\mu{}^\nu"] \arrow[dr, "\tilde{Q}"] & \\ {} & \text{(super) conformal descendant operators} & {} \end{tikzcd}}$$

Generically, can act w/ 8 Q's. For certain short & semi-short "BPS" rep'ns, some combinations of Q's annihilate.

Collection of local op's forms intricate "OPE algebra", with $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ generically singular as $x_1 \to x_2$.

Higgs branches (algebraic perspective)

In an abstract SCFT, we encounter $\mathcal{M}_H$ as a complex algebraic variety through its coordinate ring.

- Unitarity bounds imply for any operator $\Delta \geq 2R$.

- Operators with $\Delta = 2R$ necessarily have $r = j_1 = j_2 = 0$. These are superconformal primaries in $\hat{\mathcal{B}}_R$ multiplets. Annihilated by Q^1_α & $\tilde{Q}^1_{\dot\alpha}$ (so ½-BPS).

- These operators have non-singular OPEs; form a commutative, associative $\mathbb{C}$-algebra. Higgs chiral ring $\mathcal{R}_H$.

- In Lagrangian theories (conjecturally in general) $\mathcal{R}_H \cong \mathbb{C}[\mathcal{M}_H]$.

*Here symplectic structure of $\mathcal{M}_H$ is not manifest. Only algebraic derivation I know is very complicated.

Higgs branches (algebraic perspective)

Gauging realises HK quotient algebraically.

- ▷ G-symmetry $\Longrightarrow$ conserved current $\Longrightarrow$ $\hat{B}_1$ multiplet $\Longrightarrow$ cx. moment map $\mu_{\mathbb{C}} = \mu_1 + i\mu_2$ as primary.

- ▷ Upon gauging, $\mu_{\mathbb{C}} \sim \delta_{Q_-^1} \lambda_+^1$. Further restricting to gauge invariants

$$R_H^G \cong \left(R_H / \langle \mu_{\mathbb{C}} \rangle \right)^G$$

- ▷ This reproduces HK quotient b/c symplectic quotient is GIT quotient.

$$\mathrm{spec}\left(R_H^G\right) \cong \mu_{\mathbb{C}}^{-1}(0) /\!/ G \cong \vec{\mu}^{-1}(0) / G$$

An unnatural construction?

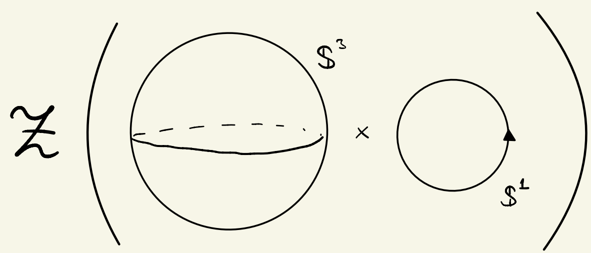
Interesting algebraic structures in SUSY QFT often arise as categorifications of numerical invariants computed by path integral.
(e.g., BPS partition functions, superconformal indices, ...)

For R_H, obvious decategorification is Hilbert series:

$$I_H(\tau) = \sum_{n=0}^{\infty} \dim\left(R_H^{(n)}\right)\tau^n \qquad R_H^{(n)} = \left\{ \begin{array}{l} R\text{-charge homogeneous subspace} \\ \text{of } R_H \text{ of degree } n. \end{array} \right\}$$

Though well-defined, does our SCFT "want" us to study this quantity? Is it computable as a path integral?

The place to look is in the world of superconformal indices.



counts states in Hilbert space assigned to S^3
≅ local operators by operator/state correspondence.

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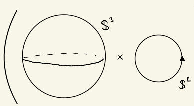
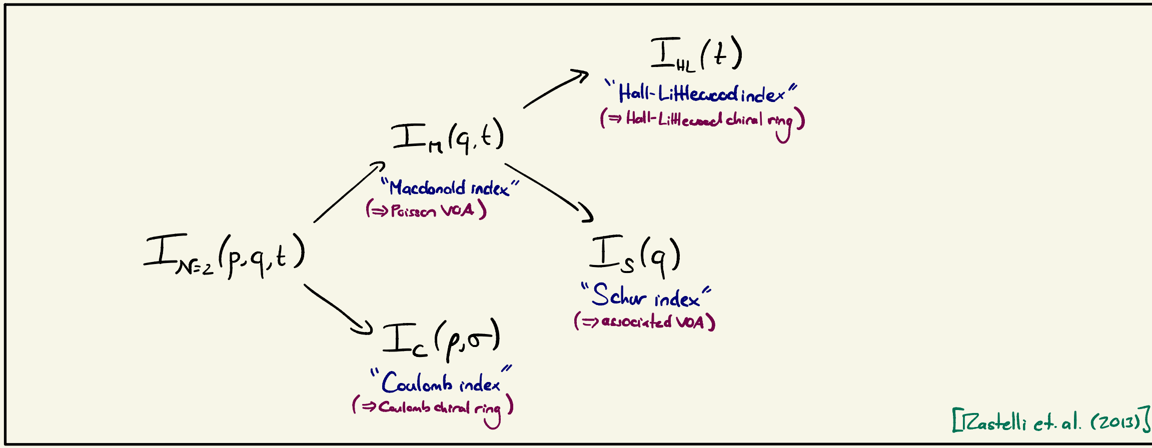
$\mathcal{Z}\Big($  $\Big)$

$$= I_{\mathcal{N}=2}(p,q,t;\underline{x}) = \mathrm{STr}_{\mathcal{H}[S^3]}\left(p^{\frac{1}{2}\tilde{\delta}_{1\dot{+}}}\, q^{\frac{1}{2}\delta_{1\dot{-}}}\, t^{R-r}\prod_{i=1}^{\mathrm{rank}(G_F)} x_i^{f_i}\right) \qquad \left[\begin{array}{l}\tilde{\delta}_{1\dot{+}} = \Delta - 2j_2 - 2R + r \\ \tilde{\delta}_{1\dot{-}} = \Delta + 2j_2 - 2R + r\end{array}\right]$$

An unnatural construction?

Generically, $I_{\mathcal{N}=2}$ counts $\frac{1}{8}$-BPS states/operators up to cancellations. In special limits counts only more highly susy operators.



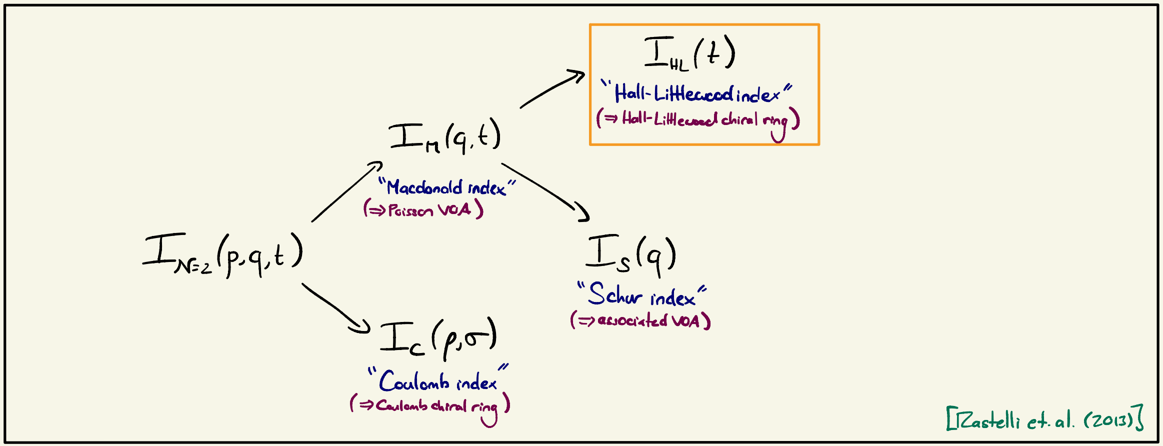
No index limit is guaranteed to return the Hilbert series of R_H.

This is in contrast to 3d $\mathcal{N}=4$ theories. So one way out is to reduce to 3d & flow to IR. See [Razamat & Willett (2014)].

Alternatively, R_H isn't the "right" object to study in 4d.

An unnatural construction?

Generically, $I_{N=2}$ counts $1/8$-BPS states/operators up to cancellations. In special limits counts only more highly SUSY operators.



I_{HL} is almost counting Higgs chiral ring operators, but not quite:

trace over states with $\Delta = 2R - r\,,\ j_2 = 0$

$$I_{HL}(t) = \mathrm{STr}_{HL}\, t^{R-r} \prod_{i=1}^{\mathrm{rank}\, G_F} x_i^{f_i}$$

The Hall-Littlewood Chiral Ring

I_{HL} counts (w/signs) operators obeying several conditions:

- $E = 2R - r \;\; (\geq 2R)$
- $j_2 = 0$

$\Big\}$ characterises $\mathcal{N}=1$ chiral ring operators

- $j_1 = -r$ → additional condition giving specialisation of chiral ring

It's an exercise in superconformal repⁿ theory to find where such states can occur.

- $\hat{B}_R$ multiplets (primary obeys $\Delta = 2R$ & $r = j_1 = j_2 = 0$) $\Longrightarrow$ superconformal primary is Higgs chiral ring operator
- $\bar{D}_{R(j_1,0)}$ multiplets (primary obeys $\Delta = 2R + j_1 + 1$ & $j_1 = -r-1$ & $j_2 = 0$) $\Longrightarrow$ Q^1_+- descendant is counted. Annihilated by $\tilde{Q}^1_{\dot{\alpha}}$, Q^1_-

Some intuition comes from (free) Lagrangian gauge theories.

hypermultiplet scalars.

$$\bar{D}_{\frac{j+n}{2}(j,0)} \longleftrightarrow P_n(q)\, \lambda^1_+ \cdots \lambda^1_+$$

positive-helicity, right-handed gauginos.

homogeneous degree-n polynomial.

The Hall-Littlewood Chiral Ring

These $\frac{3}{8}$-BPS operators have non-singular collisions and define (super-)commutative associative $\mathbb{C}$-algebra R_{HL}.

Structural properties:

- $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ graded by $(R, -r)$ w/ Grassmann parity $= r \pmod 2$.
- $(R_{HL})^{r=0} \cong R_H$
- Enriched with $(-1,0)$-Poisson bracket (extending symplectic P.B. on R_H). *
- Coupling-independent (like R_H). *

*These latter two properties follow from a slightly elaborate realisation of R_{HL} starting with VOA structure.

R_{HL} for gauge theories

For a gauge theory, R_{HL} is determined classically (like Higgs branch).

Suppose we know R_H for a theory $\mathcal{T}$, want to gauge G-symmetry of $\mathcal{T}$.

$$
\begin{array}{ccccc}
\overset{\text{original theory}}{\mathcal{T}} & \longrightarrow & \overset{\text{gauged at zero coupling}}{\mathcal{T}_0^{\mathcal{G}}} & \longrightarrow & \overset{\text{fully gauged}}{\mathcal{T}^{\mathcal{G}}} \\
R_{HL}[\mathcal{T}] & \longrightarrow & \left(R_{HL}[\mathcal{T}] \otimes R_{HL}[\text{vector}]\right)^{G} & \xrightarrow[\text{recombination}]{\text{multiplet}} & R_{HL}[\mathcal{T}^{\mathcal{G}}]
\end{array}
$$

Nontrivial step is to account for recombination of $\hat{\mathcal{B}}$ & $\bar{\mathcal{D}}$ operators at non-zero coupling. Only possible recombination patterns are as follows:

$$
\begin{tikzcd}
 & \mathcal{D}_{R+1(0,0)} \arrow[dr, "\tilde{Q}^1_-"] & \\
\hat{\mathcal{C}}_{R(0,0)} \arrow[ur, "Q^1_-"] \arrow[dr, "\tilde{Q}^1_-"'] & & \hat{\mathcal{B}}_{R+2} \\
 & \bar{\mathcal{D}}_{R+1(0,0)} \arrow[ur, "Q^1_-"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \hat{\mathcal{C}}_{R+\frac{1}{2}(j,-\frac{1}{2},0)} \arrow[dr, "\tilde{Q}^1_-"] & \\
\hat{\mathcal{C}}_{R(j,0)} \arrow[ur, "Q^1_-"] \arrow[dr, "\tilde{Q}^1_-"'] & & \bar{\mathcal{D}}_{R+\frac{3}{2}(j,-\frac{1}{2},0)} \\
 & \bar{\mathcal{D}}_{R+1(j,0)} \arrow[ur, "Q^1_-"'] &
\end{tikzcd}
$$

Recombination encoded in cohomology of (corrected) Q^1_- action on zero-coupling ring.

$\mathcal{R}_{HL}$ for gauge theories

This defines a Hall-Littlewood cohomology problem. Can formalize abstractly.

- $\mathcal{R}_{HL}[G\text{-vector-multiplet}] \cong \{\text{Grassmann algebra generated by } (\lambda^1_+)^{a=1,\dots,\dim G}\} \cong \Lambda^\bullet \mathfrak{g}$

- $HL^\bullet \cong \left(\mathcal{R}_{HL}(\mathcal{T}) \otimes \Lambda^\bullet \mathfrak{g}\right)^{\mathfrak{g}}$

(When $\mathcal{R}_{HL}[\mathcal{T}] \neq \mathcal{R}_{H}[\mathcal{T}]$, cohomological grading on $HL^\bullet$ is sum of $\Lambda^\bullet\mathfrak{g}$ grading & $(-r)$-grading on $\mathcal{R}_{HL}$.)

Differential determined by Q^1_- SUSY transformations of interacting gauge theory. Extends uniquely as differential on $HL^\bullet$.

- $d_{HL}(x) = 0$ for $x \in \mathcal{R}_{HL}(\mathcal{T})$
- $d_{HL}(\lambda^{1,a}_+) = \mu^a \in \mathcal{R}_H(\mathcal{T})$

Corresponding cohomology is our objective: $\mathcal{R}_{HL}[\mathcal{T}^{\mathfrak{g}}] \cong H^*(HL^\bullet, d_{HL})$

(Poisson bracket inherited from $\mathcal{R}_{H}[\mathcal{T}]$; λ's Poisson commute with everything.)

Relations to Koszul Homology/BRST

This cohomological story can be related to some familiar constructions.

- Before restricting to G-invariants, we have the Koszul complex/Koszul homology associated to the moment map.

- Tempting to introduce ghosts in $\Lambda^\bullet \mathfrak{g}^*$ and complete classical BRST complex of Kostant-Sternberg. This isn't what physics dictates. If so inclined, we have cohomology of relative BRST complex defined by Poisson kernel of anti-ghosts.

- In contrast to plain vanilla applications of BRST in physics, we are especially interested in higher cohomology classes.

- This description of R_{HL} for gauge theories is natural in derived algebraic geometry. It is the coordinate ring of the derived symplectic quotient of the (pre-gauging) Higgs branch by $G_{\mathbb{C}}$.

$$R_{HL}[T^{\mathfrak{g}}] \cong {}^{d}\mathcal{M}_H[T] /\!/ G_{\mathbb{C}} \qquad {}^{d}\mathcal{M}_H[T^{\mathfrak{g}}] \cong \mathrm{spec}\left(R_{HL}[T^{\mathfrak{g}}]\right)$$

Relations to Koszul Homology/BRST

(Non-)vanishing properties of Koszul homology of a ring R associated to elements $\{\mu_i \in R\}$:

- $H_0(K_\bullet(\{\mu_i\})) \cong R/\langle\{\mu_i\}\rangle$
- $\{\mu_i\}$ form regular sequence in R $\Longrightarrow$ $H_i(K_\bullet(\{\mu_i\})) = 0$ for $i \geq 1$

Non-vanishing higher homology $\Longrightarrow$ $\{\mu_i\}$ don't form regular sequence $\overset{*}{\Longrightarrow}$ $\mu_{\mathbb{C}}^{-1}(0)$ not a complete intersection.

By nature of HK quotient:

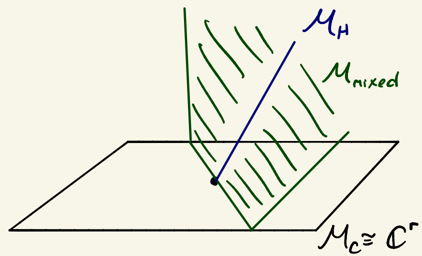
$$\left\{ \dim_{\mathbb{C}}(\mu_{\mathbb{C}}^{-1}(0)) = \dim_{\mathbb{C}}(\mathcal{M}_H) - \dim(G) + n \right\} \longleftrightarrow \left\{ \begin{array}{l} H \subset G \text{ of dimension } n \text{ stabilizes generic} \\ \text{point in } \mu_{\mathbb{C}}^{-1}(0). \end{array} \right\}$$

So higher HL cohomology detects unbroken gauge symmetry on $\mathcal{M}_H$.

* scheme-theoretic subtlety here.

Shadows of the full moduli space?

Previous result suggests that there is higher HL cohomology when there is an enhanced Higgs branch.



$\mathcal{M}_H$ embedded smoothly* in higher dimensional mixed branch: $\mathcal{M}_H \overset{i}{\hookrightarrow} \mathcal{M}_{mixed}$

R_{HL} may see something of this ambient mixed branch. There is even a natural conjecture:

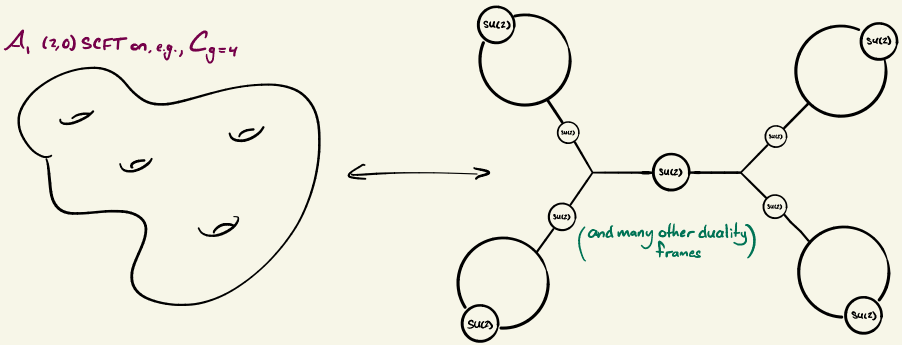
$$R_{HL} \overset{?}{\cong} \mathbb{C}\left[\Pi(N_i)\mathcal{M}_H\right]^*$$

This appears to be almost correct. What we know comes from examples.

* In general, this conjecture is probably not well-defined here.

Examples in class S

A_1 (2,0) SCFT on, e.g., $C_{g=4}$



a, b, c = hypermultiplet in (2,2,2) of $SU(2)_a \times SU(2)_b \times SU(2)_c$

a —(SU(2))— b = gauging diagonal subgroup of $SU(2)_a \times SU(2)_b$

▷ $\mathcal{M}_H \cong \mathbb{C}^2 / D_{g+1} \cong \mathbb{C}[x,y,z] / \langle y^2 = x^2 z + 4 z^g \rangle$

$$\left[D_{n+2} : \mathbb{C}^2 \to \mathbb{C}^2 ,\ \begin{pmatrix} x \\ y \end{pmatrix} \overset{z}{\mapsto} \begin{pmatrix} y \\ -x \end{pmatrix} ,\ \begin{pmatrix} x \\ y \end{pmatrix} \overset{r}{\longmapsto} \begin{pmatrix} \omega x \\ \omega^{-1} y \end{pmatrix} ,\ \omega = e^{\frac{\pi i}{n}} \right]$$

▷ residual $U(1)^g$ gauge symmetry on $\mathcal{M}_H$.

$$\left[\begin{aligned} \text{Naively, } \dim_{\mathbb{H}}(\mathcal{M}_H) &= 4 \times n_{vert} - 3 \times n_{gauge\ grps} \\ &= 4(2g-2) - 3(3g-3) \\ &= 1 - g \end{aligned} \right]$$

true dimension residual symmetries

Examples in class S

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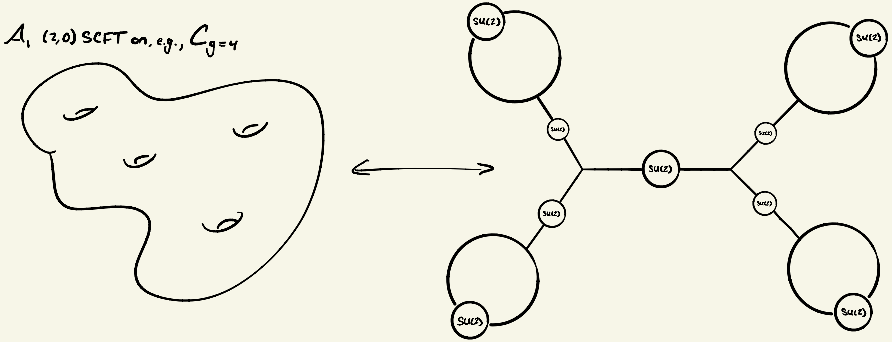
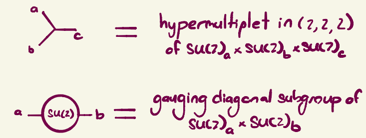
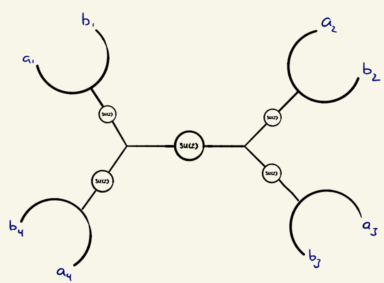
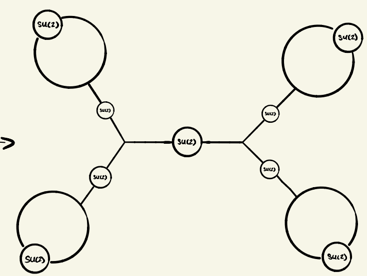




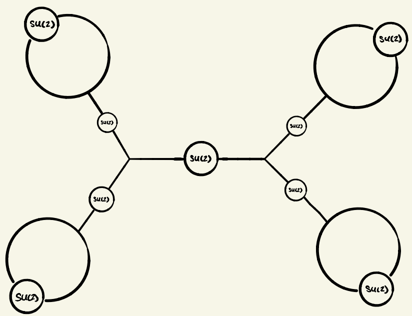
HL cohomology calculation somewhat elaborate; simplified by gauging SU(2) factors in two stages



$$\longrightarrow \quad R_{HL}^{(\text{int.})} \cong R_{H}^{(\text{int.})} \quad \longrightarrow$$



Examples in class S



Observations

- H^1_{HL} decomposes as direct sum of g identical, indecomposable modules over $H^0_{HL} \cong R_H$.
- $H^i_{HL} \neq 0$ for $i = 1, \ldots, g$; $H^i_{HL} = 0$ for $i > g$
- "Outer" $U(g)$ symmetry that leaves $H^0_{HL} \cong R_H$ invariant. Suggestive of 3d Coulomb branch symmetries.

Consider $\mathbb{C}^{2|g} / D_{g+1}$ where $D_{n+2} : \mathbb{C}^{2|g} \to \mathbb{C}^{2|g}$

$$\begin{pmatrix} x \\ y \\ \theta_i \end{pmatrix} \xmapsto{Z} \begin{pmatrix} y \\ -x \\ -\theta_i \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ \theta_i \end{pmatrix} \xmapsto{Y} \begin{pmatrix} \omega x \\ \omega^{-1} y \\ \theta_i \end{pmatrix}$$

corresponds to naive guess from previous slide.

omit all invariants constructed only from θ_i's, as well as $\theta_{i_1} \ldots \theta_{i_n} (xy)^m$ where $m < n$. Result matches R_{HL} exactly.

Conclusions

- Higgs branches of 4d $\mathcal{N}=2$ SCFTs arise naturally as derived algebraic spaces. Captured by $\mathbb{R}_{HL}[\mathcal{T}]$.
- When derived structure is included, extra symmetries emerge. These seem to be related to symmetries of 3d Coulomb branch obtained by S^1-reduction.
- Though undiscussed here, $\mathbb{R}_{HL}$ plays a prominent role in the vertex operator algebras associated to these same 4d theories.

Open questions

- Importance of conformal invariance?
- Why is 4d the right place to see this derived structure?
- Clarify relationship to full moduli space geometry.
- Better uses of DAG formalism?

Thanks For Your Attention!